PROJECTIVE CLASSIFICATION OF CUBIC SURFACES MODULO 2.

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1. A preliminary classification is given by the number N of real points (i. e., with integral coördinates) on the surface. This number N is always odd (§ 2), while there are surfaces with 1, 3, $\cdots$, 15 real points. We need examine only the non-equivalent sets of N points, equivalence being with respect to linear homogeneous transformation with integral coefficients modulo 2. In treating the surfaces whose real points are those of a fixed set of N points, the automorphs of that set are the transformations available for the specialization of the parameters in the coefficients of the surface.

For each type* of surface without singular points, all of the real straight lines are given; the number of such lines is 15, 9, 5, 3, 2, 1 or 0 (in contrast with the corresponding numbers 27, 15, 7 or 3 for cubic surfaces in ordinary space).

But the classification includes all cubic surfaces, not a cone and not formed in part of a plane.†

2. The general cubic form in x, y, z, w is

$$S = ax^3 + by^3 + cz^3 + dw^3 + lx^2y + fxy^2 + gx^2z + hxz^2 + ix^2w + jxw^2 + ky^2z + \lambda yz^2 + my^2w + nyw^2 + pz^2w + qzw^2 + rxyz + sxyw + txzw + vyzw.$$

Making use of the abbreviations

$$A = l + f, \quad B = g + h, \quad C = i + j, \quad D = k + \lambda,$$
$$E = m + n, \quad F = p + q,$$

* For certain types the configuration of the real and imaginary lines has been considered by the writer in Proc. Nat. Acad. Sciences, April, 1915.

† We do not list the quaternary cubic forms $C(x, y, z, w)$ with a real linear factor or those equivalent to a ternary form $\phi(X, Y, Z)$. A necessary condition for the equivalence of C and ϕ under the linear transformation in which the coefficients of W in x, y, z, w are α, β, γ, δ is

$$0 \equiv \frac{\partial \phi}{\partial W} = \alpha \frac{\partial C}{\partial x} + \beta \frac{\partial C}{\partial y} + \gamma \frac{\partial C}{\partial z} + \delta \frac{\partial C}{\partial w}$$

and hence that the first partial derivatives of C be linearly dependent. Attention will be called to the cases in which they are dependent and a special examination made (note that ϕ has the singular point 4). For the detection of a linear factor, note that xQ has as singular points all the intersections of $x = 0$, $Q = 0$.

we see that the values of S at the fifteen real points* in space are

$1 = (1000) : a \qquad 2 = (0100) : b \qquad 3 = (0010) : c \qquad 4 = (0001) : d$
$5 = (1100) : a + b + A \qquad 6 = (1010) : a + c + B$
$7 = (1001) : a + d + C \qquad 8 = (0110) : b + c + D$
$9 = (0101) : b + d + E \qquad 10 = (0011) : c + d + F$
$11 = (1110) : a + b + c + A + B + D + r$
$12 = (1101) : a + b + d + A + C + E + s$
$13 = (1011) : a + c + d + B + C + F + t$
$14 = (0111) : b + c + d + D + E + F + v$
$15 = (1111) : a + b + c + d + A + B + C + D + E + F + r + s + t + v.$

The sum of these linear functions is congruent to zero modulo 2. If N be the number of real points on $S \equiv 0$, N of these functions are congruent to zero and $15 - N$ are congruent to unity. Adding, we see that $0 \equiv 15 - N \pmod 2$. Hence *the number of real points on any cubic surface is odd.*

3. $N = 15$. Let the surface contain all fifteen real points. Then the linear functions in the table of § 2 are all congruent to zero, whence $a = b = c = d = A = B = C = D = E = F = r = s = t = v = 0 \pmod 2$. Excluding the case in which each coefficient of S is zero, we may set $l = 1$, after permuting the variables. Replacing y by $y + gz + iw$, we have $l = 1$, $g = i = 0$. Then replacing x by $x + kz + mw$, we have also $k = m = 0$. If $p = 0$, we have the binary form $x^2y + xy^2$ (three planes). If $p = 1$, we have

$$xy(x + y) + zw(z + w), \tag{1}$$

which has no singular point; just 15 of its 27 straight lines are real, while each of its 45 sets of 3 coplanar lines are concurrent.

4. $N = 1$. The single real point on the surface can be transformed into 1. Then, a, A, B, C, r, s, t are zero, while b, c, d, D, E, F, v are unity. We exclude the case $l = g = j = 0$, since S is then free of x. Permuting y, z, w, we may set $l = 1$. Replacing y by $y + gz + jw$, we have $l = 1$, $g = j = 0$. Replacing x by $x + kz + mw$, we have also $k = m = 0$. Replacing z by $z + pw$, we have also $p = 0$, and get

$$y^3 + z^3 + w^3 + x^2y + xy^2 + yz^2 + yw^2 + zw^2 + yzw. \tag{2}$$

There is no singular point and, of course, no real straight line.

* We shall need a list of the 35 sets of 3 collinear real points:

1	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3
2	3	4	8	9	10	14	3	4	6	7	10	13	4	5	7	9	12
5	6	7	11	12	13	15	8	9	11	12	14	15	10	11	13	14	15

4	4	4	4	5	5	5	5	6	6	6	7	7	8	8	9	10
5	6	8	11	6	7	10	13	7	9	12	8	11	9	12	11	11
12	13	14	15	8	9	15	14	10	15	14	15	14	10	13	13	12.

5. $N = 13$. We may take 1 and 2 as the two real points not on the surface. Then a, b, B, C, D, E, t, v are unity and c, d, A, F, r, s zero. Thus

$$S = x^3 + y^3 + l(x^2y + xy^2) + gx^2z + (g+1)xz^2 + ix^2w + (i+1)xw^2 + ky^2z + (k+1)yz^2 + my^2w + (m+1)yw^2 + p(z^2w + zw^2) + xzw + yzw.$$

For each of the following transformations, we give the values of the altered coefficients:

$$\begin{array}{llll} x' = x + z: & g' = g + 1, & k' = k + l, & p' = p + i + 1; \\ x' = x + w: & i' = i + 1, & m' = m + l, & p' = p + g + 1; \\ y' = y + z: & k' = k + 1, & g' = g + l, & p' = p + m + 1; \\ y' = y + w: & m' = m + 1, & i' = i + l, & p' = p + k + 1. \end{array}$$

First, let $l = 0$. Then we can make $g = i = k = m = 0$. If $p = 0$,

$$S = (x + y)(x^2 + xy + y^2 + z^2 + zw + w^2).$$

If $p = 1$, we replace y by $y + x$ and w by $w + y$ and get

$$xy(x + y) = w(w + z)(y + z). \tag{3}$$

It has no singular point. Of its 27 straight lines, the only real ones are the nine obtained by equating to zero a factor of each member of (3). Just 13 of the 45 sets of three coplanar lines are concurrent.

Second, let $l = 1$. As above, we may set $g = i = 0$. Since

$$(zw) : (gi)(km); \quad z' = z + w : i' = i + g, \quad m' = m + k,$$

we may set $k = 0$, $m = 1$, unless $k = m = 0$. In the latter case, we replace x by $x + y$ and get a form free of y. In the former case, we apply the product of the above transformations $x' = x + z$, $y' = y + z$, and obtain a form having $g = i = k = 0$, $m \doteq l = 1$, as before, but $p + 1$ in place of p. Hence we may take $p = 0$. Replacing x by $x + y$, we get

$$x^3 + xz^2 + xw^2 + y^2w + yw^2 + xzw, \tag{4}$$

for which (1110) is the only singular point. It contains only ten straight lines, all real; three coplanar lines concur if and only if one of them belongs to the pair giving the complete intersection of (4) with $w = 0$. See end of § 8.

6. Let the surface contain only three real points and let the three be not collinear. They can be transformed into 1, 2, 3. Then $a, b, c, C,$

E, F, r are zero, d, A, B, D, s, t, v are unity. Permuting x and y if necessary, we have $l = 1$, whence

$$S = w^3 + x^2y + gx^2z + (g+1)xz^2 + j(x^2w + xw^2) + ky^2z + (k+1)yz^2 + m(y^2w + yw^2) + p(z^2w + zw^2) + xyw + xzw + yzw.$$

First, let $k = 0$. Interchanging x and z, we get a like S with $k' = 0$, $g' = g + 1$, so that we may set $g = 0$. Replacing x by $x + \alpha w$, y by $y + \beta w$, we get a like S with $j' = j + \beta$, $m' = m$, $p' = p + \alpha + \beta$. Hence we may set $j = p = 0$, and get

$$w^3 + x^2y + xz^2 + yz^2 + m(y^2w + yw^2) + xyw + xzw + yzw, \tag{5}$$

with no singular point if $m = 1$, and 2 as the only one if $m = 0$.

Second, let $k = 1$. If $g = 1$, we apply (xzy) and have an S with $k = 0$, just considered. If $g = 0$, replace x by $x + pw$, y by $y + jw$, z by $z + mw$; we get

$$w^3 + x^2y + xz^2 + y^2z + xyw + xzw + yzw, \tag{6}$$

with just three singular points: $(z^2, z + z^2, z, 1)$, $z^3 + z + 1 = 0$.

Evidently there is no real line on these surfaces.

7. Let the three collinear points 1, 2, 5 be the only real points on the surface. Then c, d, F are unity and the others zero, so that

$$S = z^3 + w^3 + l(x^2y + xy^2) + g(x^2z + xz^2) + j(x^2w + xw^2) + k(y^2z + yz^2) + m(y^2w + yw^2) + pz^2w + (p+1)zw^2.$$

We make use of the transformations

$$(K)\quad x' = x + \alpha z + \beta w : k' = k + \alpha l,\quad m' = m + \beta l,\quad p' = p + \alpha j + \beta g;$$

$$(L)\quad y' = y + \gamma z + \delta w :\quad g' = g + \gamma l,\quad j' = j + \delta l,\quad p' = p + \gamma m + \delta k;$$

$$(M)\quad z' = z + \epsilon w :\quad j' = j + \epsilon g,\quad m' = m + \epsilon k,\quad p' = p + \epsilon.$$

If $l = 1$, we make $g = j = 0$ by L, $k = m = 0$ by K, $p = 0$ by M, and get

$$z^3 + w^3 + zw^2 + x^2y + xy^2. \tag{7}$$

An evident imaginary transformation on z, w replaces (7) by (1).

Next, let $l = 0$. If g, j, k, m are all zero, we make $p = 0$ by M and have the binary form $z^3 + w^3 + zw^2$. In the contrary case, we permute the variables and have $g = 1$, noting that

$$(xy) : (gk)(jm);\quad (zw) : (gj)(km)(p, p+1).$$

Then we make $j = 0$ by M, $p = 0$ by K, $k = 0$ by use of $x' = x + y$. Then if $m = 0$, S is free of y; while, if $m = 1$, we have

$$z^3 + w^3 + x^2z + xz^2 + y^2w + yw^2 + zw^2. \tag{8}$$

Neither (7) nor (8) has a singular point. They are not equivalent since the line $z = w = 0$ through our three real points is on (8) but not on (7).

8. Let the only real points on the surface be five coplanar points. They can be transformed into the points 1, 2, 3, 5, 6, viz., all but two of $w = 0$. Then d, D, v are unity and the others zero, so that

$$\begin{aligned} S = w^3 + l(x^2y + xy^2) + g(x^2z + xz^2) + j(x^2w + xw^2) + ky^2z \\ + (k + 1)yz^2 + m(y^2w + yw^2) + p(z^2w + zw^2) + yzw. \end{aligned}$$

We make use of the transformation

$$(yz) : (lg)(k,\ k + 1)(m,\ p),$$

which interchanges the two lines 125, 136, and the following transformations, which leave each line fixed:

$$\begin{array}{llll} (\alpha) & x' = x + z: & k' = k + l, & p' = p + j; \\ (\beta) & x' = x + w: & m' = m + l, & p' = p + g; \\ (\gamma) & y' = y + w: & j' = j + l, & p' = p + k + 1; \\ (\delta) & z' = z + w: & j' = j + g, & m' = m + k; \\ (\epsilon) & x' = x + y: & k' = k + g, & m' = m + j. \end{array}$$

First, let g and l be not both zero. By use of (yz), we may set $l = 1$. Then by α, β, γ, we may set $k = m = j = 0$. If $g = 1$, we apply the product $\delta\gamma$ and have $k = m = j = 0$, $g = l = 1$, $p' = p + 1$, so that we may set $p = 0$ and get

$$w^3 + x^2y + xy^2 + x^2z + xz^2 + yz^2 + yzw, \tag{9}$$

with no singular point. If $g = 0$, $p = 0$ or 1, we have

$$w^3 + x^2y + xy^2 + yz^2 + yzw, \tag{10}$$

$$w^3 + x^2y + xy^2 + yz^2 + z^2w + zw^2 + yzw, \tag{11}$$

the first having 6 as the only singular point, and the second having none. There are only three lines on (10): $w = y = 0$ and $y = w$, $z = x + bw$, $b^2 + b + 1 = 0$.

Second, let $g = l = 0$. By use of (yz), we may make $k = 0$. By use of γ, we may make $p = 0$. If $j = 0$, S is free of x. If $j = 1$, we use ϵ to make $m = 0$ and get

$$w^3 + x^2w + xw^2 + yz^2 + yzw, \tag{12}$$

having 2 as the only singular point.

It remains to prove that (9) and (11), with no singular point, are not equivalent; likewise for (10) and (12), with a single singular point. Of our five real points, 1, 2, 5 and 1, 3, 6 alone are collinear. The line $z = w = 0$ through 1, 2, 5 is on (12), but not on (9), (10) or (11). The line $y = w = 0$ through 1, 3, 6 is on (10), (11), (12), but not on (9). There are only ten lines on (12), of which the two just given are the only real ones. We can derive (12) from (4) by the imaginary transformation replacing x by y, z by $z + y$, and y by $x + y + lw$, where $l^2 + l + 1 = 0$.

9. If five real points are not coplanar, four of them can be transformed into 1, 2, 3, 4, and, by permuting the variables, the fifth can be transformed into $k = 5$, 11 or 15. Of the resulting sets $S_k = [1, 2, 3, 4, k]$, no four points of S_{15} are coplanar, while 1, 2, 3, 11 is the only coplanar set in S_{11}. In S_{11} and S_{15} there is no set of three collinear points, while in S_5 the only such set is 1, 2, 5.

An automorph of the set S_5 must permute the collinear points 1, 2, 5 and hence also the remaining points 3, 4. The only transformations leaving fixed 3, 4 and the line $z = w = 0$ (of 1, 2, 5) are the binary transformations on x, y only. Next, 3 and 4 are interchanged by (zw), which leaves 1, 2, 5 fixed.

Let the points of set S_5 be the only real points on the cubic surface. Then a, b, c, d, A, t, v are zero, and the others unity. Thus

$$\begin{aligned} S = l(x^2y + xy^2) + gx^2z + (g + 1)xz^2 + ix^2w + (i + 1)xw^2 + ky^2z \\ + (k + 1)yz^2 + my^2w + (m + 1)yw^2 + pz^2w + (p + 1)zw^2 \\ + xyz + xyw. \end{aligned}$$

The automorphs of our set of fine points are generated by

$$\begin{gathered} (xy) : (gk)(im); \qquad (zw) : (gi)(km)(p,\, p + 1); \\ x' = x + y : \quad k' = k + g + 1, \qquad m' = m + i + 1; \\ y' = y + x : \quad g' = g + k + 1, \qquad i' = i + m + 1. \end{gathered}$$

First, let g, k, m, i be not all unity. Since they are permuted transitively by the above interchanges of variables, we may set $g = 0$. Then by use of $x' = x + y$ we may set $k = 1$. If $i = 0$, $m = 1$, we make $p = 0$ by use of (zw), and get

$$(13) \qquad l(x^2y + xy^2) + xz^2 + xw^2 + y^2z + y^2w + zw^2 + xyz + xyw,$$

with no singular point if $l = 1$ and the only singular points 1 and 5 if $l = 0$. If $m = 0$, we make $i = 0$ by use of $y' = y + x$. By use of the product of $x' = x + y$ by (zw), we obtain a like form with $p' = p + 1$ and hence make $p = 0$ and get

$$(14)\qquad l(x^2y + xy^2) + xz^2 + xw^2 + y^2z + yw^2 + zw^2 + xyz + xyw,$$

with the singular point 1 if $l = 0$, none if $l = 1$. If $i = m = 1$, we have

$$(15)\qquad l(x^2y + xy^2) + xz^2 + x^2w + y^2z + y^2w + pz^2w + (p + 1)zw^2 + xyz + xyw,$$

with the singular point 4 if $p = 1$, none if $p = 0$.

Second, let $g = k = m = i = 1$. We make $p = 0$ by use of (zw) and get

$$(16)\qquad l(x^2y + xy^2) + x^2z + x^2w + y^2z + y^2w + zw^2 + xyz + xyw,$$

with the singular point 3 if $l = 1$, and 3, $(1y00)$, $y^2 + y + 1 = 0$, if $l = 0$.

The line $z = w = 0$ of the only three collinear points 1, 2, 5 is on any of our surfaces having $l = 0$, and not on those with $l = 1$. For $l = 0$, (14) and (15), with $p = 1$, are not equivalent, since their singular points are 1 and 4 respectively. For $l = 1$, (16) and (15), with $p = 1$, are not equivalent since their real non-singular points other than 1, 2, 5 are 4 and 3 respectively, and the tangent to the first at 4 is $z = 0$ which contains 1, 2, 4, 5, but not 3, while the tangent to the second at 3 is $w = x$ which contains 2 and 3, but not 1, 4 or 5.

In view of these facts and the number of singular points, it remains only to prove that, for $l = 1$, (13), (14), (15), for $p = 0$, are not equivalent, these being the only cases in which $l = 1$ and there is no singular point. The tangent plane to (15) at 3 is $x = 0$, which contains only 2, 3, 4 of our five real points; that at 4 is $z = 0$, which contains only 1, 2, 4, 5. The tangent plane to (13) at 4 is $z = x$, containing only 2 and 4; that to (14) at 4 is $x + y + z = 0$, containing only 4 and 5. The tangent plane to (13) or (14) at 3 is $x = 0$. Thus the tangent planes to (13) at 3 and 4 intersect in the line $x = z = 0$, containing only 2 and 4; while those to (14) intersect in $x = 0$, $y = z$, containing only 4.

10. Let 1, 2, 3, 4, 15 be the only real points on the surface. Then A, B, C, D, E, F are unity and the others zero. Permuting x, y and z, w, we may set $l = p = 0$. Then $(xz)(yw)$ replaces S by a like function with $g' = g + 1$. Hence we may set $g = 0$ and get

$$S = xy^2 + xz^2 + ix^2w + (i + 1)xw^2 + ky^2z + (k + 1)yz^2 + my^2w + (m + 1)yw^2 + zw^2$$

If $m = 0$, we interchange y with z and obtain a like S with $m' = 0$, $k' = k + 1$, so that we may set $k = 0$ and get

$$(17)\qquad xy^2 + xz^2 + ix^2w + (i + 1)xw^2 + yz^2 + yw^2 + zw^2,$$

with the single singular point 1 if $i = 0$, none if $i = 1$.

For $i = 0$, we replace y by $y + z$ and then x by $x + y$, w by $w + y$, and get

$$xw^2 = y^3 + yz^2 + z^3. \tag{17'}$$

The only lines on this surface are the six in $x = 0$ or $w = 0$. Although the partial derivative with respect to w is identically congruent to zero, the surface is not a cone.

If $m = 1$, we add x to y, z, w in S and get

$$k(xy^2 + xz^2 + y^2z) + (k + 1)(x^2z + x^2y + yz^2) + ixw^2 + (i + 1)x^2w + y^2w + zw^2.$$

First, let $k = 0$ and interchange x and y. If $i = 1$, we interchange y and z and get (17) for $i = 1$. If $i = 0$, we have S with $m = i = k = 1$, treated next. Second, let $k = 1$; we have S with $m = k = 1$, $i' = i + 1$. Hence we may set $i = 0$. Applying (ywz), we get (17) for $i = 0$.

11. Let 1, 2, 3, 4, 11 be the only real points on the surface. Then a, b, c, d, s, t, v are zero and the others unity. Interchanging x and y, we may set $l = 1$. Then $x' = x + z$, $y' = y + z$ replaces S by a like form with $g' = g + 1$. Hence we set $g = 1$. Applying (yz), we may set also $k = 1$. We get

$$x^2y + x^2z + ix^2w + (i + 1)xw^2 + y^2z + my^2w + (m + 1)yw^2 + pz^2w + (p + 1)zw^2 + xyz. \tag{18}$$

Denote it by $[imp]$. The cases in which there is no singular point are [101], [001], [011]; the second is transformed into the first by $x' = x + y$, $y' = x$, $z' = x + z$, and the third into the first by $x' = y$, $y' = x + y$, $z' = y + z$. For [111] there is the single singular point 4, which is invariant. For [100], [110] and [010], the only singular point is 3, and no two of the surfaces are equivalent since an automorph of our set of five points leaving 3 fixed is the identity or (xy). Finally, for [000] the partial derivative with respect to w is zero identically; replacing z by $x + y + z$ and w by $w + x + y$, we get

$$x^3 + x^2y + y^3 + xyz + zw^2.$$

It contains only nine straight lines, all imaginary,

$$w = \frac{1}{l^2}x + \frac{1}{l}y, \quad z = l^4x + l^2y; \qquad y = lx, \quad z = 0;$$

$$y = lx, \quad w = (l^2 + l)x \qquad (l^3 + l + 1 = 0).$$

The only singular points are 3 and $(x\, 1\, x^2\, x^2 + x + 1)$, $x^3 + x^2 + 1 = 0$.

12. Let the points 1, 2, 3, 5, 6, 8, 11 of the plane $w = 0$ be the only real points on the surface. Then d is the only letter not zero, so that

$$S = w^3 + l(x^2y + xy^2) + g(x^2z + xz^2) + j(x^2w + xw^2) + k(y^2z + yz^2) + m(y^2w + yw^2) + p(z^2w + zw^2).$$

We employ the transformations (leaving w fixed)

$$(\alpha) \qquad x' = x + w: \quad m' = m + l, \quad p' = p + g;$$

$$(\beta) \qquad x' = x + y: \quad k' = k + g, \quad m' = m + j;$$

$$(\gamma) \qquad x' = x + z: \quad k' = k + l, \quad p' = p + j;$$

$$(\delta) \qquad y' = y + z: \quad g' = g + l, \quad p' = p + m;$$

$$(\epsilon) \qquad y' = y + w: \quad j' = j + l, \quad p' = p + k.$$

First, let l, g, k be not all zero. In view of

$$(xy): (gk)(jm); \qquad (yz): (lg)(mp),$$

we may set $l = 1$. We make $m = k = g = j = 0$ by use of α, γ, δ, ϵ. If $p = 0$, S is free of z. If $p = 1$, we have

$$xy(x + y) = w(w^2 + wz + z^2). \tag{19}$$

Since its real straight lines must lie in $w = 0$, there are just three. The surface is obtained from (1) by replacing z by $z + bw$, w by $z + b^2w$, where $b^2 + b + 1 = 0$.

Second, let $l = g = k = 0$. If, j, m, p are all zero, $S = w^3$. In the contrary case, we may set $j = 1$ and make $m = p = 0$ by β, γ. Then S is binary.

13. Let the only real points on the surface be six coplanar points (in $w = 0$) and a point $(abc1)$ not in $w = 0$. Adding aw, bw, cw to x, y, z, we have the plane $w = 0$ and point 4. The missing point in $w = 0$ can be transformed into 11, without altering w. Since 1, 2, 3, 4, 5, 6, 8 are the only real points on the surface, a, b, c, d, A, B, D are zero and the others are unity. Thus

$$S = l(x^2y + xy^2) + g(x^2z + xz^2) + ix^2w + (i + 1)xw^2 + k(y^2z + yz^2) + my^2w + (m + 1)yw^2 + pz^2w + (p + 1)zw^2 + \sigma,$$

$$\sigma \equiv xyz + xyw + xzw + yzw.$$

We may assume that g, k, l are not all zero in view of

$$(\alpha) \qquad x' = x + y + z: \quad k' = k + l + g + 1, \quad m' = m + i + 1, \quad p' = p + i + 1.$$

Then we may take $l = 1$ since

$$(xy) : (gk)(im); \qquad (xz) \sim (lk)(ip).$$

We may set $g = 0$. For, if $g = 1$, we may make $k = 0$ by α and then apply (xy).

First, let $i = 0$. Making $p = 0$ by use of α, we have

$$(20) \quad x^2y + xy^2 + xw^2 + k(y^2z + yz^2) + my^2w + (m + 1)yw^2 + zw^2 + \sigma.$$

There is no singular point if $k = 1$. For $m = 1$, $k = 0$, the only singular points are 3 and 8; for $m = k = 0$, the only one is 3. The tangent plane at 4 is $x + (m + 1)y + z = 0$; it contains 11 if and only if $m = 1$. Hence no two of the four types (20) are equivalent. The collinear triples determine the lines $z = w = 0$, $y = w = 0$, $x = w = 0$, $x + y + z = w = 0$; the first is not on (20), the second is, the third and fourth are on if and only if $k = 0$.

Second, let $i = 1$. If $k = 1$, $p = 0$, we apply (xz) and get an S with $l = 1$, $g = i = 0$, treated above. If $k = p = 1$, we have

$$(21) \quad x^2y + xy^2 + x^2w + y^2z + yz^2 + my^2w + (m + 1)yw^2 + z^2w + \sigma.$$

Its singular points are $(x010)$, where $x^2 + x + 1 = 0$, and, in case $m = 1$, also 4. Thus no type (21) is equivalent to a type (20). Next, let $k = m = 0$; applying (xy), we have an S with $l = 1, g = i = 0$, considered above. Finally, if $k = 0$, $m = 1$, we have

$$(22) \qquad x^2y + xy^2 + x^2w + y^2w + pz^2w + (p + 1)zw^2 + \sigma.$$

The only singular points are 3, 6, 8 if $p = 0$; the only one is 4 if $p = 1$. Hence no type (22) is equivalent to a type (20) or (21).

14. Consider five coplanar points and two points not in their plane. As in § 13, we may take the five to be 1, 2, 3, 5, 6, composed of the points in the two lines 125 and 136 in the plane $w = 0$, and the additional points to be 4, P. Now (yz) permutes 2 and 3, 5 and 6; $x' = x + y$ permutes 2 and 5, 8 and 11. Hence we may assume that the line joining 4 and P meets $w = 0$ at 1, 2 or 11. In the respective cases, $P = 7$, 9 or 15. The resulting three sets of seven points are considered in turn.

Let 1, 2, 3, 4, 5, 6, 7 be the only real points on the surface. Then D, E, F alone are unity, so that

$$S = l(x^2y + xy^2) + g(x^2z + xz^2) + j(x^2w + xw^2) + ky^2z$$
$$+ (k + 1)yz^2 + my^2w + (m + 1)yw^2 + pz^2w + (p + 1)zw^2.$$

If l, g, j are all zero, there is no x. In the contrary case, we may set $g = 1$ by permuting y, z, w. We make $p = 0$, $k = 1$, by use of

$$x' = x + z: \quad k' = k + l, \quad p' = p + g;$$

$$x' = x + y: \quad k' = k + g, \quad m' = m + j.$$

Since (yw) now merely permutes l, j and adds 1 to m, we may set $m = 0$. We get

$$(23) \quad S_{lj} = l(x^2y + xy^2) + x^2z + xz^2 + j(x^2w + xw^2) + y^2z + yw^2 + zw^2.$$

If $j = 1$ there is no singular point. If $j = 0$, the only singular point is 5 if $l = 0$ and 7 if $l = 1$. There are just three collinear triples of our seven real points: 1, 2, 5 on $z = w = 0$; 1, 3, 6 on $y = w = 0$; 1, 4, 7 on $y = z = 0$. The first line is on (23) if and only if $l = 0$; the second is not; the third is if and only if $j = 0$. Hence no two of the four types (23) are equivalent.

For $j = 0$, the partial derivative of (23) with respect to w is zero identically. But (23) is then not a cone. For, if so, its vertex would be 5 or 7, according as $l = 0$ or 1; but 3, 5, 11 are collinear and also 2, 7, 12, while neither 11 nor 12 is on the surface.

15. Let 1, 2, 3, 4, 5, 6, 9 be the only real points on the surface. Then C, D, F, t, v are unity and the others zero, so that

$$S = l(x^2y + xy^2) + g(x^2z + xz^2) + ix^2w + (i + 1)xw^2 + ky^2z$$
$$+ (k + 1)yz^2 + m(y^2w + yw^2) + pz^2w + (p + 1)zw^2 + xzw + yzw.$$

The collinear triples are 1, 2, 5 (on $z = w = 0$); 1, 3, 6 (on $y = w = 0$); 2, 4, 9 (on $x = z = 0$). These lines are on S if and only if $l = 0$, $g = 0$, $m = 0$, respectively.

First, let $l = 1$. We may make $i = k = 0$ by using

$$y' = y + w: \quad i' = i + l, \quad p' = p + k + 1;$$

$$x' = x + z: \quad k' = k + l, \quad p' = p + i + 1,$$

and then make $p = 0$ by use of $(xy)(zw) : (gm)(ki)(p, p + 1)$. We get

$$(24) \quad x^2y + xy^2 + g(x^2z + xz^2) + xw^2 + yz^2 + m(y^2w + yw^2) + zw^2 + xzw + yzw,$$

with the one singular point 9 if $m = 0$, none if $m = 1$.

Second, let $l = 0$, $k \neq i$. We may make $k = 1$, $i = 0$, by use of $(xy)(zw)$, and then $p = 0$ by $x' = x + z$. We get

$$(25) \quad g(x^2z + xz^2) + xw^2 + y^2z + m(y^2w + yw^2) + zw^2 + xzw + yzw.$$

The only singular points are 1 and 3 if $g = 0$; 5 if $g = 1$, $m = 0$; none if $g = m = 1$. The four partial derivatives are linearly dependent only if $g = 0$, $m = 1$; but neither 1 nor 3 is an apex in view of 1, 4, 7 and 3, 2, 8.

Third, let $l = 0$, $k = i = 0$. The case $g = 1$, $m = 0$ may be dropped in view of $(xy)(zw)$. By using $y' = y + w$, we may set $p = 0$. Hence we have

$$(26) \quad g(x^2z + xz^2) + xw^2 + yz^2 + m(y^2w + yw^2) + zw^2 + xzw + yzw, \qquad (g, m) \neq (1, 0).$$

The four partial derivatives are linearly dependent only when $g = m = 1$; but the surface is then not a cone since no one of the four singular points

$$P = \left(y^2 + y, y, 1, \frac{y^2}{1 + y}\right), \qquad y^4 + y^3 + 1 = 0,$$

is a vertex (the point collinear with P and 3 not being on the surface). For $m = 1$, $g = 0$, the only singular point is 1. For $m = g = 0$, we replace y by $y + x$, w by $w + z$, and then x by $x + z$, and get $xw^2 + z^3 + yzw$, with the infinitude of singular points $(xy00)$. The plane $w = 0$ meets the surface in the triple line $w = z = 0$ (containing the singular points), which intersects the remaining lines $z = bw$, $x = by + b^3w$ on this ruled surface.

Fourth, let $l = 0$, $k = i = 1$. We make $p = 0$ by use of $(xy)(zw)$ and get

$$(27) \quad g(x^2z + xz^2) + x^2w + y^2z + m(y^2w + yw^2) + zw^2 + xzw + yzw.$$

The only singular points are 3 and 6 if $m = 1$, $g = 0$; 5 if $m = g = 1$; $P = (0y01)$, where $y^2 + y + 1 = 0$, if $m = 0$, $g = 1$; P, 3, 6 if $m = g = 0$. Only in the last case are the partial derivatives linearly dependent and the surface is not a cone since the third of the collinear points P, 1, $(1y01)$; 3, 2, 8; 6, 2, 11 are not on the surface.

That no two surfaces (24)–(27) are equivalent follows from the number and reality of the singular points, the fact that the pair 1, 2 is fixed by each automorph of our set of seven real points, so that 5 is fixed, and the fact that l is invariant, as well as the pair g, m (in view of the initial remark on the lines on S).

16. Let 1, 2, 3, 4, 5, 6, 15 be the only real points on the surface. Then C, D, E, F, s, t are unity and the others zero, so that

$$\begin{aligned} S = l(x^2y + xy^2) + g(x^2z + xz^2) + ix^2w + (i + 1)xw^2 + ky^2z \\ + (k + 1)yz^2 + my^2w + (m + 1)yw^2 + pz^2w \\ + (p + 1)zw^2 + xyw + xzw. \end{aligned}$$

The only collinear triples are 1, 2, 5 (on $z = w = 0$) and 1, 3, 6 (on $y = w = 0$); the first line is on S if and only if $l = 0$, the second if and only if $g = 0$.

First, let $l = g = 0$. We make $k = 0$ and $m = 0$ by use of

$$(yz) : (lg)(mp)(k, k + 1),$$

$$(\beta) \qquad \begin{array}{l} x' = x + w, \quad y' = y + w, \quad z' = z + w : \quad i' = i + l + g, \\ \qquad m' = m + l + k + 1, \quad p' = p + g + k + 1. \end{array}$$

We get

$$(28) \quad ix^2w + (i + 1)xw^2 + yz^2 + yw^2 + pz^2w + (p + 1)zw^2 + xyw + xzw.$$

The only singular points are 2, 5 if $i = 1$, and 1, 2 if $i = 0$.

Second, let $l = g = 1$. We make $k = 0$ by (yz). By use of

$$(\alpha) \qquad \begin{array}{l} x' = x + y + z : \quad k' = k + l + g, \quad m' = m + i + 1, \\ \qquad\qquad p' = p + i + 1, \end{array}$$

we make $m = 0$ if $i = 0$, and get

$$(29) \quad \begin{array}{l} x^2y + xy^2 + x^2z + xz^2 + xw^2 + yz^2 + yw^2 + pz^2w \\ \qquad + (p + 1)zw^2 + xyw + xzw, \end{array}$$

with no singular point. If $i = 1$, S becomes

$$(30) \quad \begin{array}{l} x^2y + xy^2 + x^2z + xz^2 + x^2w + yz^2 + my^2w + (m + 1)yw^2 \\ \qquad + pz^2w + (p + 1)zw^2 + xyw + xzw. \end{array}$$

There is no singular point if $p = 0$; for $p = 1$, the only one is 15 if $m = 0$ and 4 if $m = 1$.

Finally, let $l = 1$, $g = 0$. We make $k = i = 0$ by α, β and get

$$(31) \quad \begin{array}{l} x^2y + xy^2 + xw^2 + yz^2 + my^2w + (m + 1)yw^2 + pz^2w \\ \qquad + (p + 1)zw^2 + xyw + xzw. \end{array}$$

The only singular point is 6 if $p = 1$; 15 if $p = m = 0$; none if $p = 0$, $m = 1$.

Our three cases are distinguished by the invariance of the pair l, g. With each case there is no further normalization, as shown by a consideration of the eight products of (yz), α, β.

17. Consider a set of 7 points no 5 of which are coplanar. Then no 3 of the 7 are collinear. For, if 3 are in a line l, each of the remaining 4 points lies in a plane with l. But only 3 planes pass through a line l. Hence at least 2 of the 4 points are coplanar with l, whereas no 5 of the 7 points are coplanar.

Certain 4 of 6 points are coplanar since, otherwise, the 20 combinations of the 6 three at a time would give more than the 15 existing planes. Hence we may assume that one of our 7 points is 4 and that four of them

are in the plane $w = 0$. Three of the latter can be transformed into 1, 2, 3, without altering 4 or $w = 0$. The only point of $w = 0$ not collinear with two of 1, 2, 3 is 11. Thus five of our points are 1, 2, 3, 4, 11. The only points not collinear with two of these five and not in $w = 0$ are 12, 13, 14. The latter are permuted transitively by the permutations of x, y, z (which do not alter our set of five points). Hence we can transform any set of 7 points no five coplanar into 1, 2, 3, 4, 11, 12, 13, so that all such sets are equivalent.

Since 3 of 5 points are collinear, an equivalent definition of such a set of 7 points is that no 3 of its points are collinear. Hence the set can be transformed into 7, 9, 10, 12, 13, 14, 15. Let these be the only real points on the cubic surface. Then, C, E, F, s, t, v are zero and the others unity. By use of (xy), $x' = x + z$, and (yz), we may set $l = g = k = 0$, and get

$$\begin{aligned}[i,\, m,\, p] = x^3 + y^3 + z^3 + w^3 + xy^2 + xz^2 + i(x^2w + xw^2) \\ + yz^2 + m(y^2w + yw^2) + p(z^2w + zw^2) + xyz.\end{aligned} \tag{32}$$

Replacing y by $y + z$ and z by y, we get $[i,\, m + p,\, m]$. In $[i,\, m,\, p]$, we replace x by $x + y + z$ and then interchange x and z; we get $[i + p,\, m + i,\, i]$. Hence $[i,\, m,\, p]$ is equivalent to [001] or [000]. The only singular points of the latter are $(y^2 + 1,\, y,\, 1,\, 0)$, where $y^3 + y + 1 = 0$; the only one of the former is 13.

18. Any set of six real points is equivalent to one of

$$S_k = [1, 2, 3, 4, 5, k], \quad k = 6, 10, 13; \qquad S = [1, 2, 3, 4, 11, 12],$$
$$\Sigma = [1, 2, 3, 5, 6, 8].$$

For, if the six are coplanar, they can be transformed into the points Σ other than 11 of $w = 0$. If only five are coplanar the set can be transformed into S_6, 4 being the only point not in $w = 0$. The case in which no four of the six points are coplanar was excluded in § 17.

It remains to discuss the case in which four, but not more than four, of the six points are coplanar. We may assume that four of the points are in $w = 0$ and that a fifth point is 4. If no three of the six points are collinear, the argument in § 17 shows that the set is equivalent to S. If three are collinear they may be taken to be 1, 2, 5 and 3 to be the fourth point in $w = 0$. The line joining 4 and the 6th point P of the set meets $w = 0$ at a point Q not 1, 2, or 5 (since five points are not coplanar); if $Q = 3$, then $P = 10$ and the set is S_{10}; if $Q = 8$ or 11, we apply (x, y) or $y' = y + x$, neither of which alters 3, 4 or the line 1, 2, 5, and obtain $Q = 6$, whence $P = 13$, and the set is S_{13}.

We are now ready to discuss surfaces containing exactly nine real

points. First, let the only real points not on the surface be the six in set Σ. Then $d = r = 0$ and the others are unity. We make $l = 0$ by (xy). First, let $k = 1$. We make $g = i = m = p = 0$ by use of (xz), $x' = x + w$, and $y' = y + w$, and $y' = y + w$, $z' = z + w$, obtaining

$$x^3 + y^3 + z^3 + xy^2 + xz^2 + xw^2 + y^2z + yw^2 + zw^2 + xyw + xzw + yzw. \tag{33}$$

The only singular point is 7. Second, let $k = 0$. If $g = 0$, we apply $x' = x + y + z$ and then (xy) and get an S with $l = g = 0$, $k = 1$, just treated. If $g = 1$, we make $i = 0$ by $x' = x + w$ and $m = 0$ by $y' = y + w$. Then if $p = 1$, we replace x by $x + z$, y by $y + z$ and get a form free of z. If $p = 0$, we have

$$x^3 + y^3 + z^3 + xy^2 + x^2z + xw^2 + yz^2 + yw^2 + zw^2 + xyw + xzw + yzw, \tag{34}$$

with no singular point and no real straight line.

19. Let 1, 2, 3, 4, 5, 6 be the only real points not on the surface. Then C, D, E, F, s, t are zero and the others unity. By use of the automorphs $x' = x + ay + bz$ and their products by (yz), we may set $l = g = 0$ and discard the case $m = 1$, $p = 0$, obtaining

$$\begin{aligned} &x^3 + y^3 + z^3 + w^3 + xy^2 + xz^2 + j(x^2w + xw^2) + k(y^2z + yz^2) \\ &\quad + m(y^2w + yw^2) + p(z^2w + zw^2) + xyz + yzw, \quad (m, p) \neq (1, 0). \end{aligned} \tag{35}$$

No two of these types are equivalent. If $m = p = k = 0$, $j = 1$, we replace w by $w + x + y$, y by $y + z$ and get $xy^2 + w(y^2 + yz + z^2 + wy + w^2)$. Replacing x by $x + w$, z by $z + w$, we get $xy^2 + wz(z + y)$, a ruled surface with the directrices $x = w = 0$, $y = z = 0$ and ruled lines $y = \alpha z$, $w = \alpha^2 x/(\alpha + 1)$; $x = z = 0$; $x = 0$, $y = z$. Its singular points are $(x00w)$. For the remaining cases, the only singular points are

for $m = p = 0$: 15 if $k = 1$, $j = 0$; 7 if $k = j = 1$; 15, $(01zz + 1)$, $z^2 + z + 1 = 0$, if $k = j = 0$;

for $m = p = 1$: 15 if $j = k = 0$; 12, 13 if $j = 0$, $k = 1$; 9, 10, 11 if $j = 1$, $k = 0$; 14, $(ww + 1\,1\,w)$, $w^2 + w + 1 = 0$, if $j = k = 1$.

for $m = 0$, $p = 1$: 10 if $j = k = 1$; 11, 13 if $j = k = 0$; none if $j \neq k$.

The collinear triples are 7, 8, 15 on $y = z$, $x = w$; 7, 11, 14 on $y = z$, $x = z + w$; 8, 9, 10 on $x = 0$, $y = z + w$; 8, 12, 13 on $x = w$, $y = z + w$; 9, 11, 13 on $x = z$, $y = z + w$; 10, 11, 12 on $x = y$, $z = y + w$. For $m = 0$, $p = 1$, the first line is not on (35), the others are on if $j = 0$, $k = 1$,

$k = 0$, $k = j$, $k = j$, respectively; in particular, with just two real lines if $j = 0$, $k = 1$.

20. Let 1, 2, 3, 4, 5, 10 be the only real points not on the surface. Then a, b, c, d, A, F are unity and the others zero. We make $l = p = 0$ by use of (xy) and (zw); and get

$$\begin{aligned}[gjkm] = x^3 + y^3 + z^3 + w^3 + xy^2 + g(x^2z + xz^2) \\ + j(x^2w + xw^2) + k(y^2z + yz^2) + m(y^2w + yw^2) + zw^2.\end{aligned} \tag{36}$$

The only transformations permuting forms of this type are generated by

$$x' = y, \quad y' = x + y; \quad (xz)(yw); \quad z' = w, \quad w' = z + w.$$

Under these, [0000] is invariant; [1001], [0111], [1110] are permuted; [0110], [1011], [1101] are permuted; while the remaining nine are permuted. Hence there are four non-equivalent types (36). Type [1000] has the one singular point 6, dependent derivatives, but is not a cone. Type [1001] has the singular points 6 and $(x\ 1\ x + 1\ 1)$, x arbitrary; by the transformation

$$X = x + y + w, \quad Y = y + w, \quad Z = x + z, \quad W = x + z + w,$$

[1001] equals $XY^2 + ZW^2$, with the singular points $(X0Z0)$; it is a ruled surface with directrices $x = z = 0$, $y = w = 0$, and ruled lines $z = d^2x$, $y = dw$, and $x = w = 0$. Type [0110] has no singular point and the single real line $z = x$, $y = w$. Type [0000] has no singular point; the only three real lines on it are $z = x, y = w; y = z = x + w; w = x, z = x + y$; an evident imaginary transformation, cogredient in x, y and z, w, transforms it into (1).

21. Let 1, 2, 3, 4, 5, 13 be the only real points not on the surface. Then a, b, c, d, A, v are unity and the others zero. By use of $x' = x + y$, we make $l = 0$ and get

$$\begin{aligned}x^3 + y^3 + z^3 + w^3 + xy^2 + g(x^2z + xz^2) + j(x^2w + xw^2) \\ + k(y^2z + yz^2) + m(y^2w + yw^2) + p(z^2w + zw^2) + yzw.\end{aligned} \tag{37}$$

First, let $g = j = 1$. If $k = m = 1$, the only singular points are 8, 9, 15 if $p = 0$, and $(10w^2w)$, where $w^2 + w + 1 = 0$, if $p = 1$. If $km = 0$, we make $m = 0$ by (zw), and $k = 0$ by

$$(\alpha) \quad x' = x + z, \quad w' = w + z: \quad g' = g + j + 1,$$
$$k' = k + m + 1, \quad p' = p + j + 1.$$

Then the singular points are 6, 7, 15 if $p = 1$ and $(0yy^21)$, where $y^2 + y + 1 = 0$, if $p = 0$.

Second, let $gj = 0$. We make $j = 0$ by (zw) and then $g = 0$ by α. In view of (zw), we may interchange k and m. For $k = m = 0$, the only singular points are $(yyy^2 1)$, $y^3 = 1$, if $p = 0$, and none if $p = 1$. For $k = 1$, $m = 0$, the only singular points are 10 and 11 if $p = 1$, none if $p = 0$. For $k = m = 1$, the only singular points are 11 and 12 if $p = 0$, 15 if $p = 1$.

The resulting types (37) having the same number of real and imaginary singular points are seen to be not equivalent either by the incidence of these points with the invariant plane $y = 0$ or by the intersections of the latter with the surface.

The types (37) with no singular point have $j = g = m = 0$, and either $k = 0$, $p = 1$ or $k = 1$, $p = 0$. For the former the three real lines on the surface are $y = z$, $x + w = 0$ or z, and $y = w$, $x = z + w$. The last is the only real line in the second case.

22. Let 3, 6, 9, 10, 12, 13 be the only real points not on the surface. Since no three of the six points are collinear, the set is equivalent to S of § 18. Then c, D, E, v are unity and the others zero. We get

$$
(38)\qquad \begin{aligned} &z^3 + l(x^2y + xy^2) + g(x^2z + xz^2) + j(x^2w + xw^2) + ky^2z \\ &\qquad + (k+1)yz^2 + my^2w + (m+1)yw^2 + p(z^2w + zw^2) + yzw. \end{aligned}
$$

We normalize by the transformations

$$
x' = x + \alpha y + \beta z + \gamma z; \qquad y' = y + z + w, \quad z' = w, \quad w' = z;
$$

and $w' = w + z$. First, let $l = 0$. If $g = j = 0$, x does not occur. In the contrary case, we may make $j = 1$, $g = p = m = 0$. Then the only singular point is 2 if $k = 0$, 11 if $k = 1$. Second, let $l = 1$. We may make $k = m = j = 0$. For $g = 1$, there is no singular point if $p = 0$ and the factor $y + z$ if $p = 1$. For $g = 0$, there is no singular point if $p = 1$, while the only one is 7 if $p = 0$. No two of the resulting types are equivalent. Those with no singular point are given by $l = 1$, $k = m = j = 0$, and $g = 1$, $p = 0$ or $g = 0$, $p = 1$; each has only three real straight lines: for the former, $y = z = 0$; $y = z$, $w = 0$; and $y = z = w$; for the latter, $y = z = 0$; $x = 0$, $y = z$; $x = y = z$.

23. Finally, consider the surfaces with exactly eleven real points. Here let 1, 2, 3, 5 be the only real points not on the surface. Then d, B, D, r, t, v are zero and the others unity. We make $l = i = m = 0$ by (xy), $x' = x + w$, and $y' = y + w$. By the product of $x' = x + y$, $y' = x$, and the last two transformations, we make $g = 0$, and get

$$
(39)\qquad \begin{aligned} &x^3 + y^3 + z^3 + xy^2 + xw^2 + k(y^2z + yz^2) + yw^2 + pz^2w \\ &\qquad + (p+1)zw^2 + xyw. \end{aligned}
$$

If $k = 0$, we make $p = 0$ by $z' = z + w$; then the only singular point is 13. For $k = 1$, the only singular points are 7, 11, 15 if $p = 1$; $(0, y, y + 1, 1)$ where $y^2 + y + 1 = 0$, if $p = 0$.

24. Let 1, 2, 3, 11 be the only real points not on the surface. Then a, b, c, C, E, F are unity and the others zero. Thus

$$S = x^3 + y^3 + z^3 + l(x^2y + xy^2) + g(x^2z + xz^2) + (j + 1)x^2w + jxw^2 + k(y^2z + yz^2) + my^2w + (m + 1)yw^2 + pz^2w + (p + 1)zw^2.$$

If $lg = 0$, we make $l = g = 0$ by $(yz) : (lg)(mp)$ and

$$(\alpha) \quad x' = x + z, \quad y' = y + z : \quad g' = g + l + 1, \quad k' = k + l + 1, \quad p' = p + j + m + 1.$$

We make $j = m = 0$ by $x' = x + w$ and $y' = y + w$. If $k = 0$, we make $p = 0$ by $z' = z + w$ and get

$$x^3 + y^3 + z^3 + x^2w + yw^2 + zw^2, \tag{40}$$

with the one singular point 14, not a vertex by 14, 15, 1. Next, let $k = 1$. If $p = 0$, we replace y by $y + z$ and obtain a form free of z. If $p = 1$, we have

$$x^3 + y^3 + z^3 + x^2w + y^2z + yz^2 + yw^2 + z^2w. \tag{41}$$

The latter has no singular point and contains exactly five real straight lines: $y = 0$, $z = x$; $y = w = x + z$; $z = 0$, $y = x + w$; $z = w$, $y = x$; $w = 0$, $x = y + z$.

Next, let $l = g = 1$. If $k = 0$, we apply $y' = y + x$, $z' = z + x$, and have $l = g = 0$, just treated. Thus let $k = 1$. Make $j = 0$ by $x' = x + w$. If $m = 0$, make $p = 0$ by α. If $m = 1$, $p = 0$, make $m = 0$ by (yz). Hence we may set $m = p$. Replacing x by $x + y + z$ and then y by $y + z$, we get a form free of z.

25. Lastly, let 11, 12, 13, 14 (in preference to 1, 2, 3, 4) be the only real points not on the surface. Then r, s, t, v alone are unity. We get

$$\begin{aligned} &l(x^2y + xy^2) + g(x^2z + xz^2) + j(x^2w + xw^2) + k(y^2z + yz^2) \\ &\quad + m(y^2w + yw^2) + p(z^2w + zw^2) + xyz + xyw + xzw + yzw. \end{aligned} \tag{42}$$

The only available transformations are the permutations of x, y, z, w. The six coefficients $l, \cdots, p$ are permuted transitively since

$$(xy) : (gk)(jm); \quad (xz) : (lk)(jp); \quad (xw) : (lm)(gp).$$

If all six are zero, the only singular points are 1, 2, 3, 4, no one being a vertex. If five are zero, we may set $l = 1$; the singular points are 3, 4.

Next, let $l = 1$ and a single other one be unity; the surface with $g = 1$ is transformed into that with k, j or m unity by (xy), (zw), $(xy)(zw)$; in the remaining case $p = 1$, $x + y + z + w$ is a factor. For $l = g = 1$, $j = k = m = p = 0$, the only singular points are 4, 7.

If $l = g = j = 1$, $k = m = p = 0$, the only singular point is 15. By use of (xy), we obtain $l = k = m = 1$. Next, if $l = g = k = 1$ or $l = j = m = 1$ (and the remaining coefficients vanish), $x + y + z + w$ is a factor. The six remaining cases in which $l = 1$ and two further coefficients are unity are derived by use of (xy), (xw), (yz), (xyw), (xzy) from the case $l = g = m = 1$. For the latter, 10 is the only singular point.

If a single coefficient is zero, we may set $l = 0$. The only singular points are then $(x100)$, $x^2 + x + 1 = 0$. If only two coefficients are zero, we may set $l = p = 0$ or $l = g = 0$. In the first case, the only singular points are $P = (x100)$, $x^2 + x + 1 = 0$, and $Q = (00z1)$, $z^2 + z + 1 = 0$, no one being a vertex since P and 3 are collinear with $(x110)$, not on the surface. For $l = g = 0$, $j = k = m = p = 1$, there is no singular point and just five real lines: $z = w = 0$; $y = w = 0$; $w = 0$, $x = y + z$; $x = 0$, $y = z + w$; $y = z$, $x = w$, of which the first three are coplanar but not concurrent, while the first and last two are coplanar and concurrent.

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